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SHORTER COMMUNICATION

AN APPROXIMATE ANALYTICAL SOLUTION FOR THE RADIATION EXCHANGE BETWEEN TWO FLAT SURFACES SEPARATED BY AN ABSORBING GAS

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INTRODUCTION

THIS note summarizes the results of an approximate analytical solution for the temperature distribution in an absorbing gas layer with no internal energy sources and bounded by two flat surfaces radiating as black bodies [1]. This result is used to obtain an explicit expression for the net radiation exchange between the surfaces in terms of the various physical parameters involved.

TEMPERATURE DISTRIBUTION

The problem posed here was treated previously by Usiskin and Sparrow [2] who obtained solutions for the required quantities by a numerical solution of the appropriate integral equation. In the system investigated it was assumed that the absorption properties of the gas could be represented by a single coefficient of absorption which was independent of temperature. On this basis they derived a linear integral equation for the "temperature" distribution function $\theta(x)$. By introducing the generalized exponential integrals $E_n(x)$, the appropriate form for the source-free equation may be written [3]

$$2\,\theta(x) = E_2[\kappa\,(1-x)] + \kappa \int_0^1 \theta(s) E_1(\kappa\,|x-s|)\,\mathrm{d}s \quad (1)$$

where

$$\theta(x) \equiv \frac{W_g(x) - \sigma T_2^4}{\sigma(T_1^4 - T_2^4)} \quad (2)$$

and $0 \leqslant x \leqslant 1$ is the fractional distance across the gas layer measured from the cold (T_2) wall in the direction of the hot (T_1) wall. The quantity $4kW_g(x)\,\mathrm{d}x$ represents the energy radiated per unit time from a gas layer of thickness $\mathrm{d}x$ and unit cross-sectional area [4]. A special case of $W_g(x)$ is the familiar expression $\sigma T_g^4(x)$, where $T_g(x)$ is the temperature of the gas. The parameter κ is the optical thickness of the gas layer and is defined $\kappa \equiv kL$; k is the absorption coefficient of the gas and L its thickness

Equation (1) is in the form of the inhomogeneous Fredholm equation of the second kind, and formal solutions can be generated by an iterative procedure. In selecting a first approximation, we are guided by our knowledge of the general methods of the transport theory [5]. A first solution which is frequently used in this approach is a linear function; this is borne out in the present case by an inspection of the Usiskin–Sparrow results. Therefore, in constructing our approximate analytical solution we introduce the form

$$\theta(x;\kappa) = a(\kappa)x + c(\kappa)$$

into the integral as a first-order approximation and calculate the resulting expression for $\theta(x;\kappa)$ at the left. If the assumed linear form is indeed a reasonable first estimate for $\theta(x)$, then the resulting expression from the left should have the form

$$\theta(x;\kappa) = a(\kappa)x + c(\kappa) + R(x;\kappa) \quad (3)$$

where the correction term $R(x;\kappa)$ will be small. The application of this procedure is straightforward, and leads easily to an expression for R. In carrying out this calculation, it is necessary to impose also two conditions on $\theta(x)$ so that the specification of the quantities $a(\kappa)$ and $c(\kappa)$ is complete. These are

$$\theta'(\tfrac{1}{2};\kappa) = \tfrac{1}{2} \qquad \theta'(\tfrac{1}{2};\kappa) = a(\kappa). \quad (4)$$

The first condition is a consequence of the antisymmetry property of the function θ [1]. The second implies that $R'(\frac{1}{2};\kappa) = 0$. The substitution of the linear form for θ into the right-hand side of equation (1) yields the result:

$$R(x;\kappa) = \tfrac{1}{2}\left\{\frac{a(\kappa)}{2\kappa}\left[e^{-\kappa x} - e^{-\kappa(1-x)}\right] - [c(\kappa) + \tfrac{1}{2}a(\kappa)x]E_2(\kappa x) + [1 - c(\kappa) - \tfrac{1}{2}a(\kappa)(1+x)]E_2[\kappa(1-x)]\right\} \quad (5)$$

with

$$a(\kappa) = \frac{\kappa}{2}\,e^{\kappa/2}\,E_1\left(\frac{\kappa}{2}\right); \quad c(\kappa) = \tfrac{1}{2}\,[1 - a(\kappa)]. \quad (6)$$

The resulting expression for $\theta(x;\kappa)$ given in equation (3), when compared with the numerical solution obtained by Usiskin and Sparrow, is found to be accurate to within 1 per cent. The comparison is especially good for large κ; when $\kappa \gg 1$, the expressions for a and c have the asymptotic forms:

$$a(\kappa) \sim 1 - \frac{2}{\kappa}; \qquad c(\kappa) \sim \frac{1}{\kappa}. \quad (7)$$

It may be observed from the form of the function (3) that further iteration will yield an integral equation for the correction term $R(x;\kappa)$. Because of the antisymmetry property of θ, the next-higher-order approximation may be constructed by introducing an x^3 term into (3). A

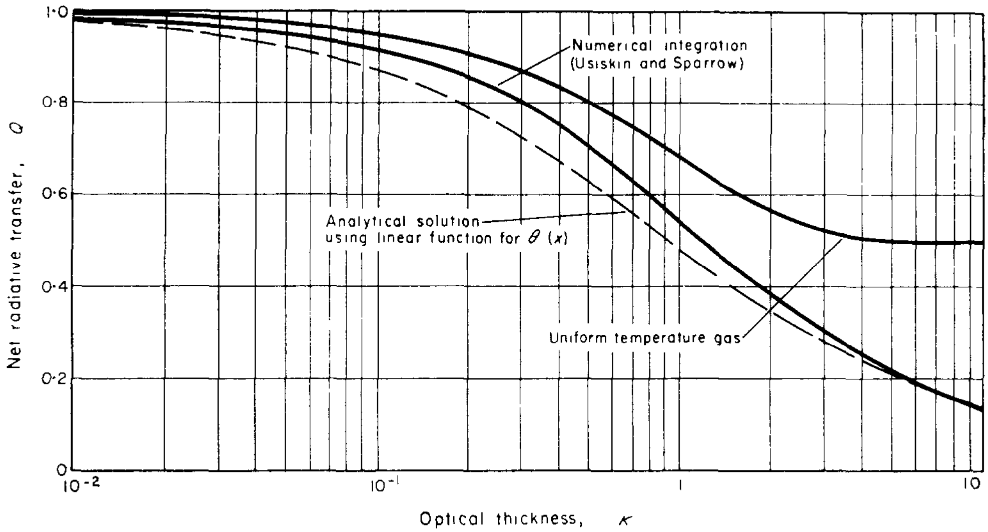

FIG. 1. Net radiative transfer to cold wall.

continuation of this procedure will generate a power series in odd x which will converge, thereby providing an increasingly accurate analytical expression for θ.

NET RADIATIVE TRANSFER

Of practical interest is the calculation of the net radiative transfer between the hot wall and gas, and the cold wall. The appropriate equation may be written in the form [1]

$$Q(\kappa) = 2E_3(\kappa) + 2\kappa \int_0^1 \theta(s)E_2(\kappa s)\,ds \qquad (8)$$

where

$$Q(\kappa) \equiv \frac{q(\kappa)}{\sigma(T_1^4 - T_2^4)} \qquad (9)$$

and $q(\kappa)$ is the net energy transferred per unit area and time to the cold wall. It is easily shown that $Q(\kappa)$ has the following properties:

$$\lim_{\kappa \to 0} Q(\kappa) = 1, \qquad \lim_{\kappa \to \infty} Q(\kappa) = 0, \qquad (10)$$

as is to be expected on physical grounds. The function $Q(\kappa)$ has been computed by Usiskin and Sparrow for the interval $(0 \leqslant \kappa \leqslant 2)$ using the numerical solution for $\theta(x)$ mentioned previously. Of particular interest to this discussion is the comparison of their result with that obtained using the analytical form for θ suggested in the preceding section. If one neglects entirely the correction term R, then direct substitution into equation (8) yields

$$Q(\kappa) = \tfrac{1}{2}\left[1 + \left(\frac{4}{3\kappa} - 1\right)a(\kappa)\right] - \frac{2a(\kappa)e^{-\kappa}}{3\kappa} + \left[1 - \frac{a(\kappa)}{3}\right]E_3(\kappa). \qquad (11)$$

It may be shown that this result also satisfies the limits (10). A graphical comparison with the exact numerical solution is shown in Fig. 1. Evidentally, the use of the linear form for θ yields an estimate for Q which agrees to within about 10 per cent in the interval $(0{\cdot}1 < \kappa < 1)$; even better agreement is obtained outside the interval.

A final comparison of some practical value may be drawn from the result obtained for the net radiation exchange $Q(\kappa)$ in a system wherein the gas layer is at some uniform equilibrium temperature [1]. In that case, it may be shown that $Q(\kappa)$ takes the form

$$Q(\kappa) = \tfrac{1}{2}(1 + e^{-\kappa}). \qquad (12)$$

This function also is shown in Fig. 1; agreement with the variable-temperature case is reasonably good up to two optical thicknesses of gas layer.

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